130 Final Exam Cheat Sheet

Heating Problem

$\frac{dT}{dt} = -k(T - T_o)$
T_o = outside temperature

Mixing Problem

$\frac{dA}{dt} = c_1 r_1 - \frac{A}{V} r_2$
$V = V_0 + (r_1 - r_2)t$
c_1, solution mixture in
r_1, in rate
r_2, out rate

Inner Product Spaces

1. $\langle v, v\rangle \geq 0$ Furthermore, $\langle v, v\rangle = 0 \leftrightarrow v = 0$
2. $\langle v, u\rangle = \langle u, v\rangle$
3. $\langle ku, v\rangle = k\langle u, v\rangle$
4. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

$||v|| = \langle v, v\rangle$
$\cos^{-1}(\frac{\langle v,u\rangle}{||v||||u||})$

Gram-Schmidt

$$v_1 = x_1$$
$$v_2 = x_2 - \frac{\langle x_2, v_1\rangle}{||v_1||^2} v_1$$
$$\vdots$$
$$v_n = x_m - \sum_{k=1}^{m-1} \frac{\langle x_m, v_k\rangle}{||v_k||^2} v_k$$

Variation of Parameters

$$F(x) = y'' + y'$$
$y_h = b_1 y_1(x) + b_2 y_2(x), y_1 y_2$ are L.I.
$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$u_1 = \int^t -\frac{y_2 F(t)dt}{w[y_1, y_2](t)}$$
$$u_2 = \int^t \frac{y_1 F(t)dt}{w[y_1, y_2](t)}$$
$$y = y_h + y_p$$

ODEs

1st Order Linear	Use integrating factor, $I = e^{\int P(x)dx}$
Separable:	$\int P(y)dy/dx = \int Q(x)$
HomogEnEous:	$dy/dx = f(x,y) = f(xt, yt)$ sub $y = xV$ solve, then sub $V = y/x$
Exact:	If $M(x,y) + N(x,y)dy/dx = 0$ and $M_y = N_x$ i.e. $\langle M, N\rangle = \nabla F$ then $\int_x M + \int_y N = F$
Order Reduction	Let $v = dy/dx$ then check other types *If purely a function of y,* $\frac{dv}{dx} = v\frac{dv}{dy}$
Variation of Parameters:	When $y'' + a_1 y' + a_2 y = F(x)$ F contains $\ln x$, $\sec x$, $\tan x$, $\div$
Bernoulli	$y' + P(x)y = Q(x)y^n$ $\div y^n$ $y^{-n}y' + P(x)y^{1-n} = Q(x)$ *Let* $U(x) = y^{1-n}(x)$ $\frac{dU}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ $\frac{1}{1-n}\frac{du}{dx} + P(x)U(x) = Q(x)$ *solve as a 1st order*
Cauchy-Euler	$x^n y^n + a_1 x^{n-1} y^{n-1} + \cdots + a_{n-1} y^{n-2} + a_n y = 0$ guess $y = x^r$
3 Cases:	
1) Distinct real roots	$y = ax^{r_1} + bx^{r_2}$
2) Repeated real roots	$y = Ax^r + y_2$ *Guess* $y_2 = x^r u(x)$ *Solve for* $u(x)$ *and choose one* $(A = 1, C = 0)$
3) Distinct complex roots	$y = B_1 x^a \cos(b \ln x) + B_2 x^a \sin(b \ln x)$

Series Solution

$y'' + p(x)y' + q(x)y = 0$
Useful when $p(x), q(x)$ not constant
Guess $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$

e^x	$\sum_{n=0}^{\infty} x^n/n!$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Systems

$\vec{x}' = A\vec{x}$	
A is diagonalizable	$\vec{x}(t) = a_1 e^{\lambda_1 t}\vec{v_1} + \cdots + a_n e^{\lambda_n t}\vec{v_n}$
A is not diagonalizable	$\vec{x}(t) = a_1 e^{\lambda_1 t}\vec{v_1} + a_2 e^{\lambda t}(\vec{w} + t\vec{v})$ where $(A - \lambda I)\vec{w} = \vec{v}$ $\vec{v}$ is an Eigenvector w/ value λ i.e. $\vec{w}$ is a generalized Eigenvector
$\vec{x}' = A\vec{x} + \vec{B}$	Solve y_h $\vec{x_1} = e^{\lambda_1 t}\vec{v_1}, \vec{x_2} = e^{\lambda_2 t}\vec{v_2}$ $\vec{X} = [\vec{x_1}, \vec{x_2}]$ $\vec{X}\vec{u}' = \vec{B}$ $y_p = \vec{X}\vec{u}$ $y = y_h + y_p$

Matrix Exponentiation

$A^n = SD^nS^{-1}$
D is the diagonalization of A

Laplace Transforms

$L[f](s) = \int_0^{\infty} e^{-sx} f(x)dx$

$f(t) = t^n, n \geq 0$	$F(s) = \frac{n!}{s^{n+1}}, s > 0$
$f(t) = e^{at}, a$ *constant*	$F(s) = \frac{1}{s-a}, s > a$
$f(t) = \sin bt, b$ *constant*	$F(s) = \frac{b}{s^2+b^2}, s > 0$
$f(t) = \cos bt, b$ *constant*	$F(s) = \frac{s}{s^2+b^2}, s > 0$
$f(t) = t^{-1/2}$	$F(s) = \frac{\pi}{s^{1/2}}, s > 0$
$f(t) = \delta(t - a)$	$F(s) = e^{-as}$
f'	$L[f'] = sL[f] - f(0)$
f''	$L[f''] = s^2 L[f] - sf(0) - f'(0)$
$L[e^{at}f(t)]$	$L[f](s - a)$
$L[u_a(t)f(t - a)]$	$L[f]e^{-as}$

Gaussian Integral

$\int_{-\infty}^{+\infty} e^{-1/2(\vec{x}^T A\vec{x})} = \frac{\sqrt{2\pi}^n}{\sqrt{\det A}}$

Complex Numbers

Systems of equations	If $\vec{w_1} = u(\vec{t}) + iv(\vec{t})$ is a solution, $\vec{x_1} = u(\vec{t}), \vec{x_2} = v(\vec{t})$ are solutions i.e. $\vec{x_h} = c_1\vec{x_1} + c_2\vec{x_2}$
Euler's Identity	$e^{ix} = \cos x + i \sin x$

Vector Spaces

$v_1, v_2 \in V$

1. $v_1 + v_2 \in V$
2. $k \in \mathbb{F}, kv_1 \in V$
3. $v_1 + v_2 = v_2 + v_1$
4. $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
5. $\forall v \in V, 0 \in V \mid 0 + v_1 = v_1 + 0 = v_1$
6. $\forall v \in V, \exists - v \in V \mid v + (-v) = (-v) + v = 0$
7. $\forall v \in V, 1 \in \mathbb{F} \mid 1 * v = v$
8. $\forall v \in V, k, l \in \mathbb{F}, (kl)v = k(lv)$
9. $\forall k \in \mathbb{F}, k(v_1 + v_2) = kv_1 + kv_2$
10. $\forall v \in V, k, l \in \mathbb{F}, (k + l)v = kv + lv$