
Methods of Applied Mathematics Problem Set 1

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January 26, 2015

1 EXERCISE 6.1

Compute the Fourier transform of $e^{-|x|}$ for $x \in \mathbb{R}$.

Solution:

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x-ix\xi} dx + \int_{-\infty}^0 e^{x-ix\xi} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (e^{-x-ix\xi} - e^{-x+ix\xi}) dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-(1+i\xi)} (-1) - \frac{1}{-1+i\xi} (-1) \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1-i\xi}{1+\xi^2} + \frac{-(1+i\xi)}{1+\xi^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{-2i\xi}{1+\xi^2} \\ &= -\sqrt{\frac{2}{\pi}} \frac{i\xi}{1+\xi^2}\end{aligned}$$

2 EXERCISE 6.2

Compute the Fourier transform of $e^{-a|x|^2}$, $a > 0$, directly, where $x \in \mathbb{R}$.

Solution:

$$\begin{aligned}
 \hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|^2} e^{-ix\xi} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x+\frac{i\xi}{2a})^2 + \frac{\xi^2}{4a}} dx \quad x' \doteq x + \frac{i\xi}{2a} \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{\infty} e^{-ax^2} dx \\
 &= \frac{e^{-\frac{\xi^2}{4a}}}{2a}
 \end{aligned}$$

3 EXERCISE 6.4

$f \in L_1(\mathbb{R}^d)$, and $f(x) = g(|x|)$ for some g , show that $\tilde{f}(\xi) = h(|\xi|)$ for some h . Solution:

$$\begin{aligned}
 \hat{f}(\xi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx \quad (\text{polar coordinate transformation}) \\
 &= \frac{1}{(2\pi)^{d/2}} \int_0^{\infty} \int_{\partial S^d} f(r\alpha) e^{-ir\alpha \cdot \xi} r^{d-1} d\alpha dr \\
 &= \frac{1}{(2\pi)^{d/2}} \int_0^{\infty} g(r) r^{d-1} \int_{\partial S^d} e^{-ir\alpha \cdot \xi} d\alpha dr \tag{3.1}
 \end{aligned}$$

For any ξ , \exists rotation Q , such that $\xi = |\xi|Qe_1$, so that $\alpha \cdot \xi = \alpha^T \xi = \alpha^T (Qe_1) |\xi| = |\xi| (Q^T \alpha)^T e_1$

$$\begin{aligned}
 \therefore (3.1) &= \frac{1}{(2\pi)^{d/2}} \int_0^{\infty} g(r) r^{d-1} \int_{\partial S^d} e^{-ir(Q^T \alpha) \cdot e_1 |\xi|} d\alpha dr \\
 &= \frac{1}{(2\pi)^{d/2}} \int_0^{\infty} g(r) r^{d-1} \int_{\partial S^d} e^{-ir(Q^T \alpha) \cdot e_1 |\xi|} d(Q^T \alpha) dr \quad \beta \doteq Q^T \alpha \\
 &= \frac{1}{(2\pi)^{d/2}} \int_0^{\infty} g(r) r^{d-1} \int_{\partial S^d} e^{-ir\beta_1 |\xi|} d\beta dr \\
 &= h(|\xi|)
 \end{aligned}$$

only depends on $|\xi|$

4 EXERCISE 6.10

Let the field be complex and define $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ by

$$Tf(x) = \int e^{-|x-y|^2/2} f(y) dy$$

Show that T is positive, injective operator, but not surjective.

Solution:

$$\begin{aligned} \text{Positive} &\iff \langle Tf(x), f(x) \rangle \geq 0, \forall f \in L^2(\mathbb{R}^d) \\ &\iff \int \int e^{-|x-y|^2/2} f(y) dy f(x) dx \geq 0, \forall f \in L^2(\mathbb{R}^d) \end{aligned}$$